Lecture 31

Colouring and Bipartite Graphs

Exam Scheduling

Suppose we want to schedule exams for the courses with following course numbers:

111, 340, 400, 250, 181, 900, 720, 910

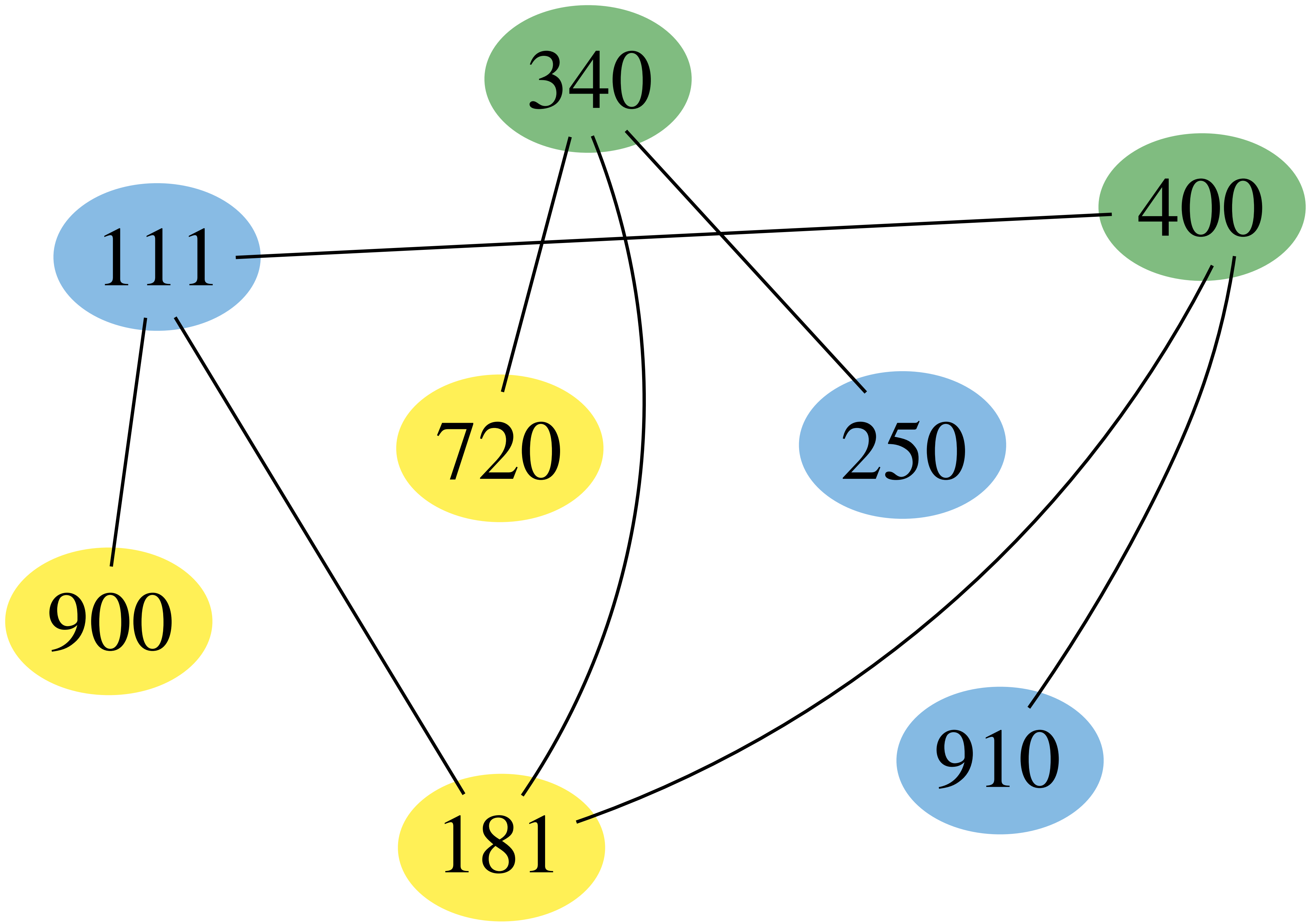
Some courses have some common students:

(111,400), (111,181), (111,900), (340,250), (340,181), (340,720), (400,181), (400,910)

How many exam slots are necessary to schedule exams so that courses with some common students have different slots?

Exam Scheduling

Have a vertex for every course and put an edge between any two courses if they have some common students.



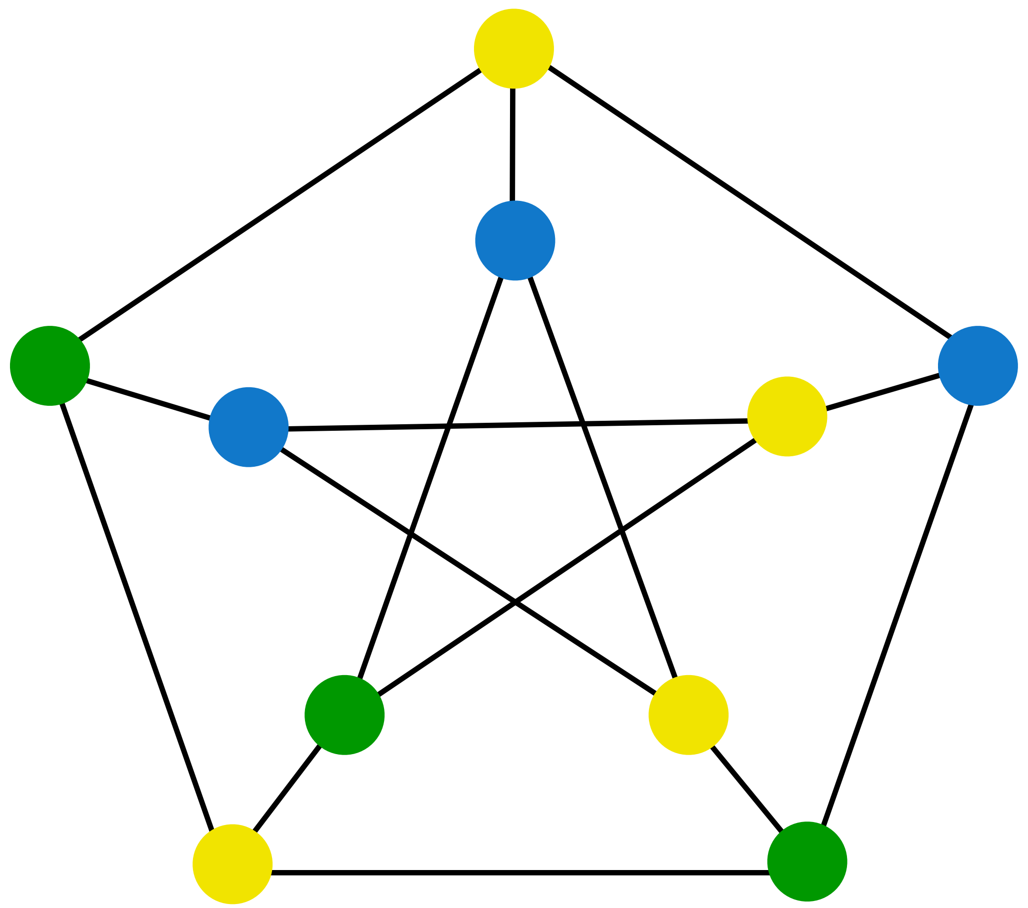
Number of required slots = *Minimum number of colours required to colour each vertex so that neighbours have different colours*

Graph Colouring

Definition: The **chromatic number** of a graph G, denoted by $\chi(G)$, is the smallest integer k for which the vertices of G can be coloured by k colours so that adjacent vertices are coloured differently.

Definition: A graph is called **k-colourable**, if its vertices can be coloured using k colours such that there are no adjacent vertices with the same colour.

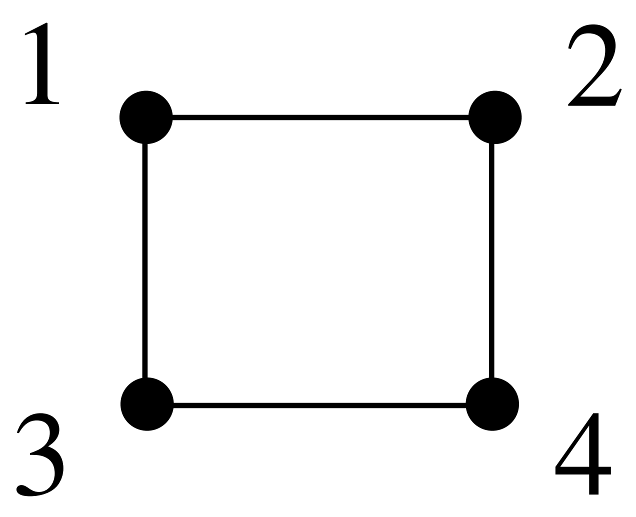
Example: The below graph is called **Petersen Graph** and its chromatic number is 3.

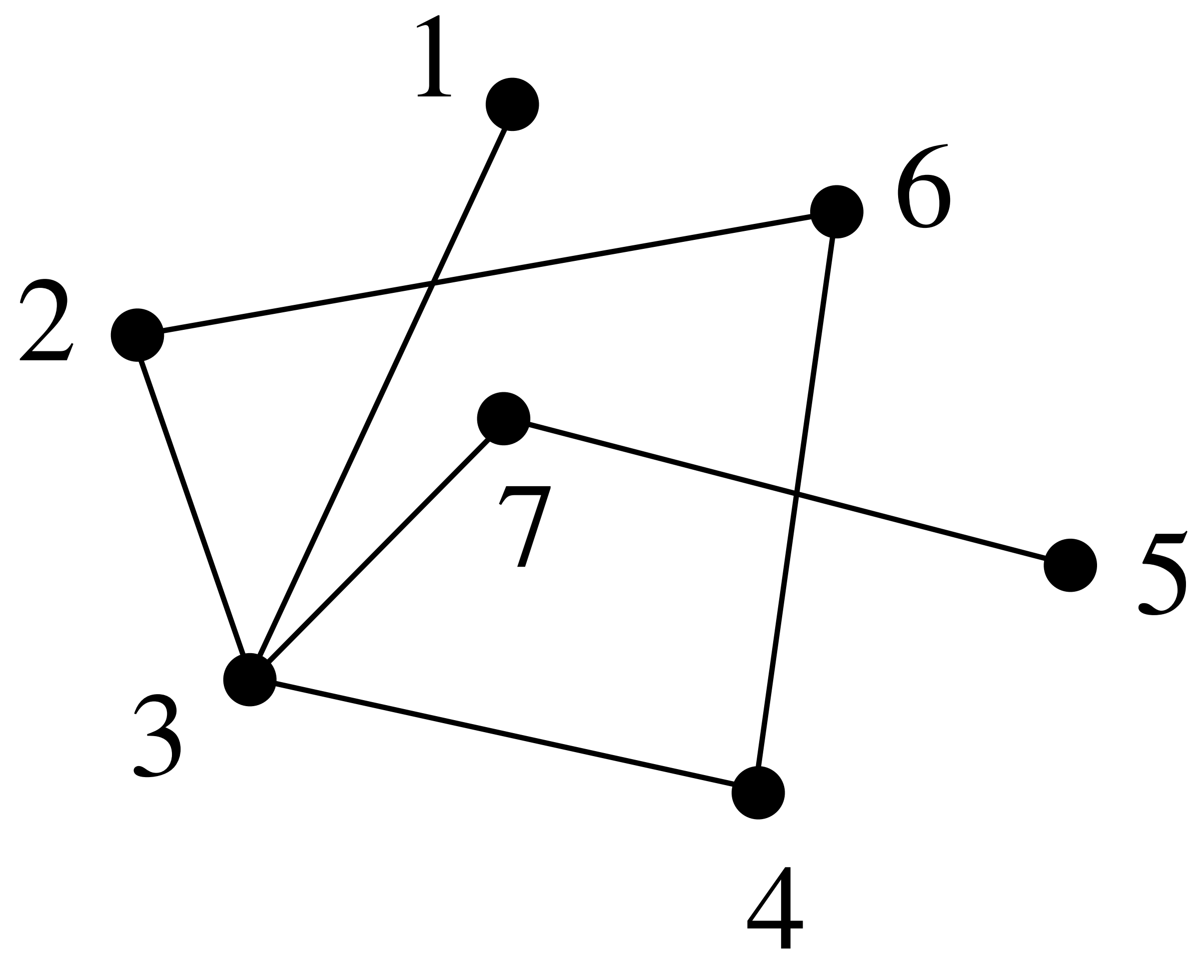


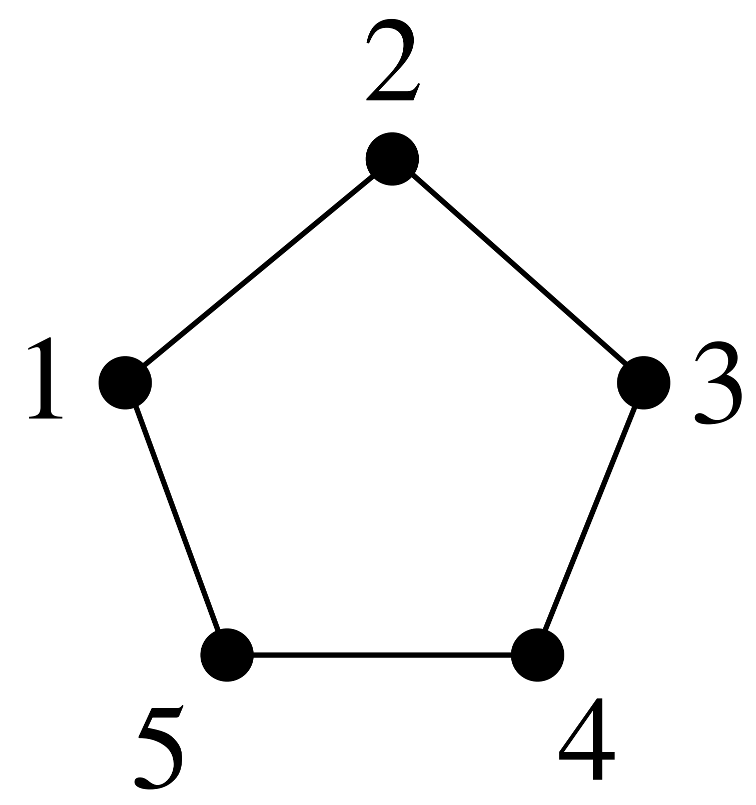
Bipartite Graphs

Definition: A graph G is **bipartite** if the vertex set of G can be split into disjoint sets A and B such that each edge of G is incident on one vertex in A and one vertex in B.

Examples:







$A = \{1,4\}, B = \{2,3\}$

$A = \{1,2,4,7\}, B = \{3,5,6\}$

Not bipartite

Observation: A graph is bipartite if and only if it is 2-colourable.

Alternative Characterisation of Bipartite Graphs

Theorem: A graph is bipartite if and only it does not contain a cycle of odd length.

Proof: ($\Longrightarrow$) Let G be a bipartite graph and hence, should be 2-colourable.

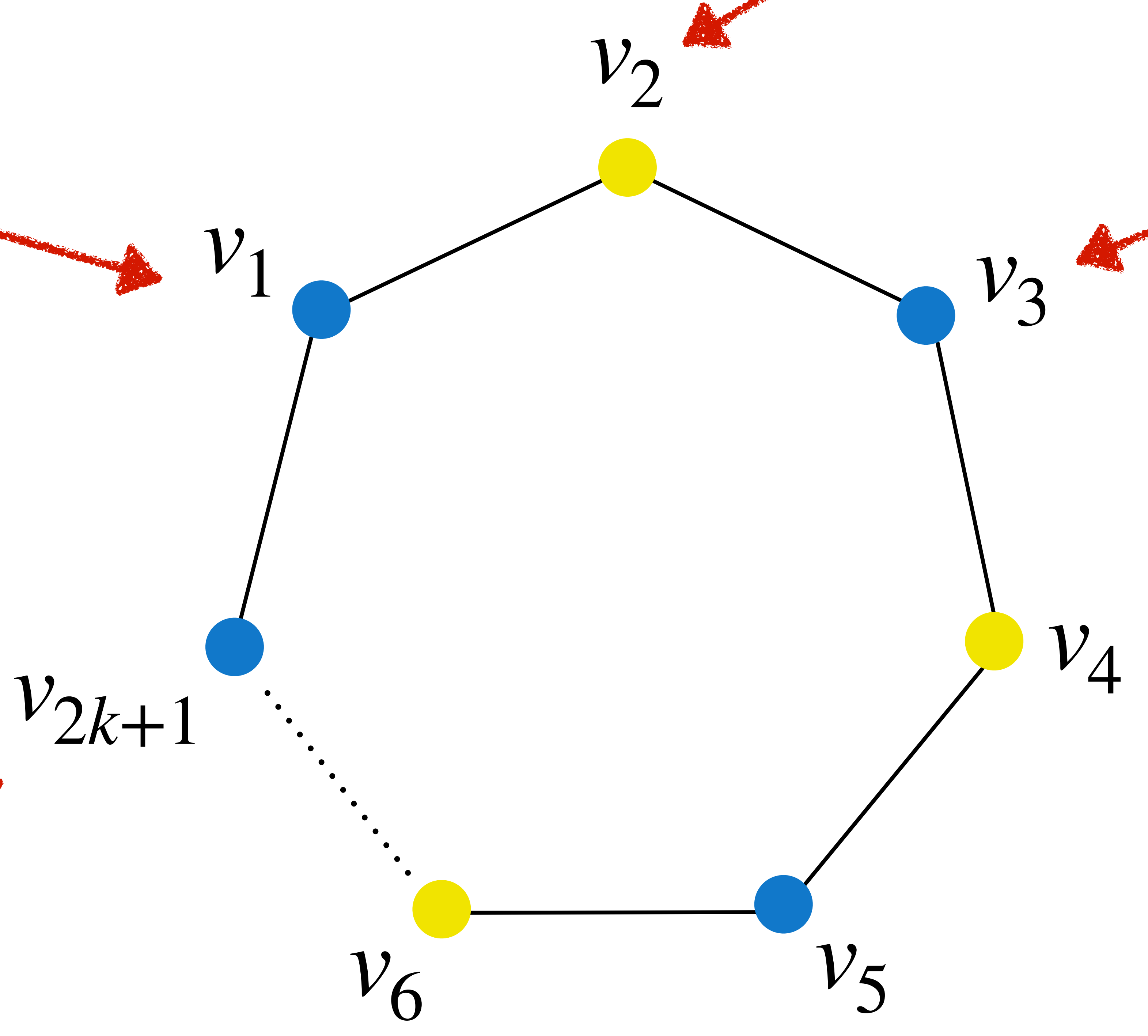
Suppose there is an odd length cycle C in G.

Try to colour vertices of C using 2 colours.

v_2 has to be coloured yellow.

WLOG, colour v_1 blue.

v_3 has to be coloured blue.



3rd colour is needed for v_{2k+1}.
Hence, G is not 2-colourable, a contradiction.

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Alternative Characterisation of Bipartite Graphs

Proof: ($\Longleftarrow$) Let G be a graph with no odd cycles.

We describe a way to colour G using 2 colours.

Pick any vertex u of G and colour it red.

Let $d(u, v)$ denote the length of a shortest path from u to v

Divide the rest of the vertices of G into 2 sets.

- $S_{even} = \{v \mid d(u, v) \text{ is even}\}$
- $S_{odd} = \{v \mid d(u, v) \text{ is odd}\}$

Colour all the vertices of S_{odd} blue and colour all the vertices of S_{even} red.

We prove now that this is a valid colouring.

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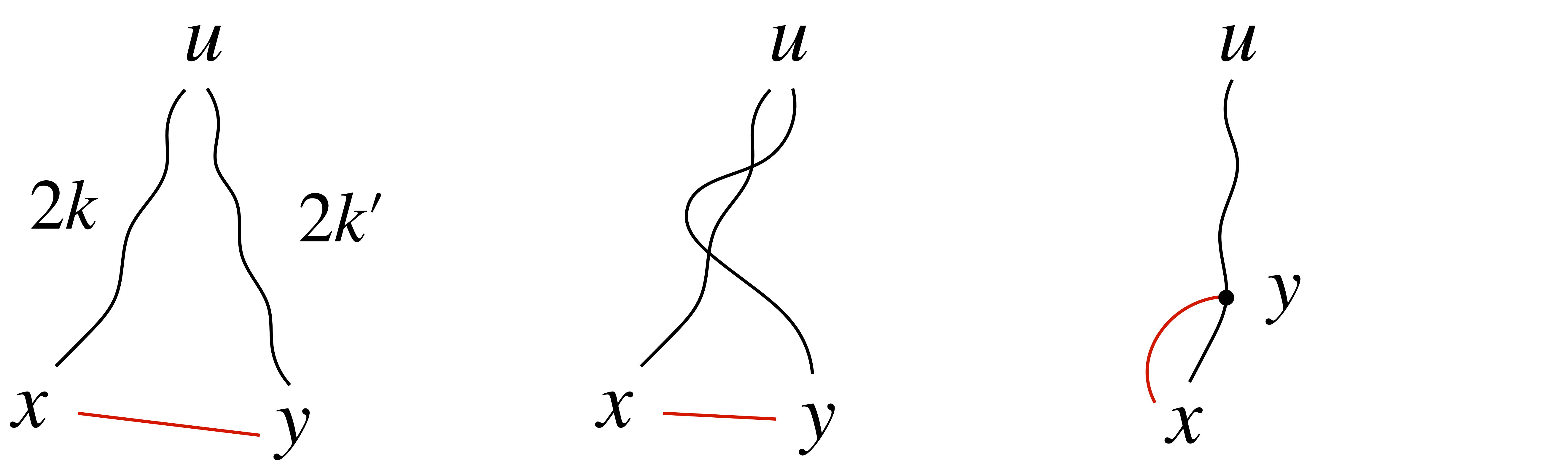
Alternative Characterisation of Bipartite Graphs

Proof: ($\Longleftarrow$) Suppose it is not a valid colouring.

Then there must be two adjacent vertices, say x and y, with the same colour, say red.

Because x and y are of same colour, they must have even length shortest path from u.

Let P be a shortest path from u to x and Q be a shortest path from u to y.



$P + xy + reverse(Q)$ forms a closed "walk" of odd length.

A closed "walk" of odd length implies a cycle of odd length in G, a contradiction. ■